Differential Geometry
Exercise Sheet 1

Solve Exercise 2, 3 and at least one between Exercise 1 and Exercise 4.

Exercise 1 The n-dimensional sphere S^n_R of radius $R > 0$ (with center in the origin 0) is defined by

$$S^n_R := \{x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1} \mid ||x|| = R\} ,$$

where $|| \cdot ||$ is the Euclidean norm.

The n-dimensionall ball of radius $R > 0$ (with center in the origin 0) is defined by

$$B^n_R := \{y = (y^1, \ldots, y^n) \in \mathbb{R}^n \mid ||y|| < R\} \subset \mathbb{R}^n .$$

1. For $j = 1, \ldots, n+1$, we consider $U^+_j := \{x \in S^n_R \mid x^j > 0\}$ and $U^-_j := \{x \in S^n_R \mid x^j < 0\}$. Define $\phi^\pm_j : U^\pm_j \to B^n_R \subset \mathbb{R}^n$ as follows:

$$\phi^\pm_j(x) = (x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^{n+1}) .$$

   prove that $\phi^\pm_j$ has a smooth inverse and $\{(\phi^\pm_j)^{-1}\}$ is a smooth atlas with $2(n+1)$ charts.

2. Let $N = (0, \ldots, 0, R) \in S^n_R$ be the North Pole. The *stereographic projection from N* is the map $\phi_N : S^n_R \setminus N \to \mathbb{R}^n$ that maps each $p = (p^1, \ldots, p^{n+1}) \in S^n_R \setminus N$ to the intersection point between the line joining N and p and the hyperplane $\{x^{n+1} = 0\} \subset \mathbb{R}^{n+1}$ (identified canonically $\{x^{n+1} = 0\}$ with $\mathbb{R}^n$).

   (a) Prove that $\phi_N : S^n_R \setminus N \to \mathbb{R}^n$ is a diffeomorphism and the following holds:

$$\phi_N(p) = \frac{R}{R - p^{n+1}}(p^1, \ldots, p^n) .$$

   Similarly, let $S = (0, \ldots, 0, -R) \in S^n_R$ be the South Pole, the *stereographic projection from S* is the map $\phi_S : S^n_R \setminus S \to \mathbb{R}^n$ that maps each $p \in S^n_R \setminus S$ to the intersection point between the hyperplane $\{x^{n+1} = 0\} \subset \mathbb{R}^{n+1}$ and the line joining S and p.

   (a) Prove that $\phi_S : S^n_R \setminus S \to \mathbb{R}^n$ is a diffeomorphism and the following holds:

$$\phi_S(p) = \frac{R}{R + p^{n+1}}(p^1, \ldots, p^n) .$$

(a) Prove that ϕ_S and ϕ_N have smooth inverses and that $\{\phi_S^{-1}, \phi_N^{-1}\}$ is a smooth atlas of charts for S_R^n.

3. Conclude from general principles that the two atlases described above are compatible.

Exercise 2 Suppose that $M_1, \ldots, M_k$ are smooth manifolds of dimensions $n_1, \ldots, n_k$ respectively. Prove that $M_1 \times \ldots \times M_k$ is a smooth manifold of dimension $n_1 + \ldots + n_k$.

Exercise 3 Which of the following linear spaces intersect transversally ? Explain your answer.

1. The xy plane and the z axis in $\mathbb{R}^3$;
2. The xy plane and the plane spanned by $\{(3,2,0), (0,4,-1)\}$ in $\mathbb{R}^3$;
3. The plane spanned by $\{(1,0,0), (2,1,0)\}$ and the y axis in $\mathbb{R}^3$;
4. $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$ (Depends on k, l, n);
5. $\mathbb{R}^k \times \{0\}$ and $\mathbb{R}^l \times \{0\}$ in $\mathbb{R}^n$ (Depends on k, l, n);
6. $\mathbb{R}^k \times \{0\}$ and the diagonal in $\mathbb{R}^k \times \mathbb{R}^k$;
7. The symmetric ($A^t = A$) and skew-symmetric ($A^t = -A$) matrices in $M(n)$.

Exercise 4 Let $n, p \in \mathbb{N}$ with $n \geq p$, and $M \subset \mathbb{R}^n$. Consider the following statements:

(a) (Local definition via straightening) For every $x \in M$ there exists a neighborhood $U \subset \mathbb{R}^n$ with $x \in U$, a neighborhood V of 0 in $\mathbb{R}^n$ and a smooth diffeomorphism $f : U \to V$ such that $f(U \cap M) = V \cap (\mathbb{R}^p \times \{0\})$.

(b) (Local definition via implicit function) For every $x \in M$ there exists a neighborhood U of $x \in \mathbb{R}^n$ and a smooth map $f : U \to \mathbb{R}^{n-p}$ which is a submersion in x and such that $U \cap M = f^{-1}(0)$.

(c) (Local definition via graph) For every $x \in M$ there exists a neighborhood $U \subset \mathbb{R}^n$ of x, an identification via a linear map $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$, an open subset $V \subset \mathbb{R}^p$ and a smooth map $f : V \to \mathbb{R}^{n-p}$ such that $U \cap M$ is the graph of f.

(d) (Local definition via parametrization) For every $x \in M$, there exists a neighborhood $U \subset \mathbb{R}^n$ if x, a neighborhood of $V \subset \mathbb{R}^p$ of 0 and a smooth map $f : V \to \mathbb{R}^n$ such that $f(0) = x$, f is an immersion in 0 and f is a homeomorphism of V in $U \cap M$.

We say that $M \subset \mathbb{R}^n$ is a *smooth submanifold* of dimension p if it verifies the statement (1). Prove that all the statements above are equivalent, following this strategy:

1. Statement (a) implies Statement (b)
2. Statement (a) implies Statement (d)
3. Statement (d) implies Statement (a)
4. Statement (b) implies Statement (a)
5. Statement (c) implies Statement (d)
6. Statement (b) implies Statement (c)

Solutions to Exercise 4 Recall the following results:

Theorem 0.1 (A.2 Local inversion Theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}^n$ be a C^k function, where $k \geq 1$. If $x \in U$ is a point where the Jacobian matrix $Df(x)$ is invertible, then there exists an open neighborhood V of x in U and an open neighborhood W of $f(x)$ in $\mathbb{R}^n$ such that $f : V \to W$ is a C^k diffeomorphism.*

(Here, the Jacobian matrix $Df(x)$ is the $n \times n$ matrix whose (i,j)-th entry is $\frac{\partial f_i}{\partial x_j}(x)$, where $f = (f_1, \ldots, f_n)$ and $x = (x_1, \ldots, x_n)$. The invertibility of $Df(x)$ means that its determinant $\det(Df(x))$ is nonzero.)

Here is the translation in English and LaTeX:

Let $p, q \leq n$ be fixed in $\mathbb{N}$, and let k be an element of $(\mathbb{N} - \{0\}) \cup \{\infty, \omega\}$. Recall that a local diffeomorphism of $\mathbb{R}^n$ at a point x_0 is a diffeomorphism from an open neighborhood of x_0 to an open neighborhood of x_0 that sends x_0 to itself. The maps $(x_1, \ldots, x_p) \mapsto (x_1, \ldots, x_p, 0, \ldots, 0)$ from $\mathbb{R}^p$ to $\mathbb{R}^n$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_q)$ from $\mathbb{R}^n$ to $\mathbb{R}^q$, and $(x_1, \ldots, x_p) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$ from $\mathbb{R}^p$ to $\mathbb{R}^q$, where $r \leq \min\{p, q\}$, are respectively an immersion, a submersion, and a map of constant rank r. The following results state that these examples are, locally and modulo local diffeomorphisms, the only ones.

Theorem 0.2 (A.5 (Local Normal Form Theorem for Immersions))**.** *Let U be an open set in $\mathbb{R}^p$ containing 0, and let $f : U \to \mathbb{R}^n$ be a C^k function that is an immersion at 0 with $f(0) = 0$. Then there exists a C^k local diffeomorphism ψ of $\mathbb{R}^n$ at 0 such that, in a neighborhood of 0, we have*

$$\psi \circ f(x_1, \ldots, x_p) = (x_1, \ldots, x_p, 0, \ldots, 0).$$

Theorem 0.3 (A.6 (Local Normal Form Theorem for Submersions))**.** *Let U be an open set in $\mathbb{R}^n$ containing 0, and let $f : U \to \mathbb{R}^q$ be a C^k function that is a submersion at 0 with $f(0) = 0$. Then there exists a C^k local diffeomorphism ϕ of $\mathbb{R}^n$ at 0 such that, in a neighborhood of 0, we have*

$$f \circ \phi(x_1, \ldots, x_n) = (x_1, \ldots, x_q).$$

Let us show that (a) implies (b). If x, U, f are as in (1), then we can assume that $f(x) = 0$. Let $f_1, \ldots, f_n$ be the components of f, and let $g : U \to \mathbb{R}^{n-p}$ be the map with components $f_{p+1}, \ldots, f_n$. Then g is a C^∞ submersion such that $g^{-1}(0) = U \cap M$.

Let us show that (a) implies (d). If x, U, V, f are as in (1), then we can assume that $f(x) = 0$. Let $W = V \cap (\mathbb{R}^p \times 0)$. Let $f^{-1}|_W : W \to U \cap M$ be the restriction of f^{-1} to $W = V \cap (\mathbb{R}^p \times \{0\})$. Then $f^{-1}|_W$ is a C^∞ immersion at 0 that sends 0 to x, and is a homeomorphism between W and $U \cap M$.

Let us show that (d) implies (a). If x, U, V, f are as in (4), then by the local normal form theorem for immersions (Theorem A.5), we can assume (after possibly restricting U and V) that there exists a C^∞ diffeomorphism $\psi : U \to W$ for some open neighborhood W of 0 in $\mathbb{R}^n$ such that $\psi \circ f(x_1, \ldots, x_p) = (x_1, \ldots, x_p, 0, \ldots, 0)$ on V. In particular, $\psi(U \cap M) = \psi \circ f(V) = W \cap (\mathbb{R}^p \times \{0\})$.

The fact that (b) implies (a) can be shown in a similar way, using the local normal form theorem for submersions (Theorem A.6).

The fact that (c) implies (d) is immediate, since if x, U, V, f are as in (3), then we can assume that $x = 0$ and $f(0) = 0$, and the map $F : y \mapsto (y, f(y))$ is then a homeomorphism from V to $U \cap M$ that is a C^∞ immersion at 0 with $F(0) = 0$.

Finally, let us show that (b) implies (c). Let x, U, f, $f_1, \dots, f_{n-p}$ be as in (2), and assume that $x = 0$. Without loss of generality, we may assume that the matrix $[\frac{\partial f_i}{\partial x_{j+p}}(0)]_{1 \le i,j \le n-p}$ (i.e. the submatrix of the Jacobian matrix of f at 0 corresponding to the last $n - p$ columns) is invertible.

Let $\mathrm{pr}_1 : \mathbb{R}^p \times \mathbb{R}^{n-p} \to \mathbb{R}^p$ be the projection onto the first factor. Let $F : U \to \mathbb{R}^p \times \mathbb{R}^{n-p}$ be the map defined by $F(y) = (\mathrm{pr}_1(y), f(y))$. Then the differential of F at 0 is invertible. Thus, by the inverse function theorem (Theorem A.2), F is a C^∞- diffeomorphism in a neighborhood of 0. Its inverse is a map of the form $y \mapsto (\mathrm{pr}_1(y), G(y))$ for some C^∞ function $G : W \subset \mathbb{R}^n \to \mathbb{R}^{n-p}$ from a neighborhood W of 0 in $\mathbb{R}^n$. Thus, after possibly restricting U, we find that $U \cap M = f^{-1}(0) = F^{-1}(\mathbb{R}^p \times \{0\})$ is the graph of G .