Differential Geometry
Exercise Sheet 2

Solve at least 2 of the following problems.

Exercise 1 Let $\Omega \subseteq \mathbb{R}^{n+m}$ be an open subset. Let $F : \Omega \to \mathbb{R}^m$ be a smooth map and

$$\mathrm{Crit}(F) := \{p \in \Omega \mid dF_p : \mathbb{R}^{n+m} \to \mathbb{R}^m \text{ is not sujective } \}$$

the set of critical points of F. A *regular value* for F is a point in $F(\Omega)$ which is not the image of a critical point.

1. Prove that $\mathrm{Crit}(F)$ is a closed subset of Ω.
2. Prove that for every $a \in F(\Omega)$, then $M_a = F^{-1}(a) \setminus \mathrm{Crit}(F)$ is a smooth manifold of dimension n, whose structure is compatible with the topology induced by $\mathbb{R}^{n+m}$.
3. Prove that if a is a regular value for F, the level set $F^{-1}(a) = \{p \in \Omega \mid F(p) = a\}$ is a smooth manifold of dimension n.
4. Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined as

$$F(x, y, z, w) = (x^2 + y, x^2 + y^2 + z^2 + w^2 + y) \ .$$

Prove that $(0, 1) \in \mathbb{R}^2$ is a regular value for F and $F^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

Exercise 2 Let V be a vector space over a field $\mathbb{K}$. Define an equivalence relation $\sim$ on $V \setminus \{O\}$ as follows: $v \sim w$ if and only if $v = \lambda w$ for some $\lambda \in \mathbb{K}^\star$. The *projective space* $\mathbb{P}(V)$ is the set of equivalence classes of $\sim$ on $V \setminus \{O\}$. Notice that $\mathbb{P}(V)$ is the set of all one-dimensional subsets of V. There is a natural projection $V \setminus \{O\} \mapsto [v]$.

1. Prove that if V is a finite-dimensional vector space over $\mathbb{R}$, then $\mathbb{P}(V)$ is a smooth manifold. What is its dimension?

2. The space $\mathbb{RP}^n := \mathbb{P}(\mathbb{R}^{n+1})$ is called the *real projective space* of dimension n. If $x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1} \setminus \{O\}$, we denote $[x] := [x^0 : \dots : x^n]$ its projection in $\mathbb{RP}^n$. For $j = 0, \dots, n$ let

$$U_j = \{[x^0 : \dots x^n] \in \mathbb{RP}^n \mid x^j \neq 0\} ,$$

with bijections $\phi_j : U_j \to \mathbb{R}^n$ defined as the following:

$$\phi_j([x^0 : \dots : x^n]) = (\frac{x^0}{x^j}, \dots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \dots, \frac{x^n}{x^j}) .$$

Prove that $\{(U_j, \phi_j)\}$ is an atlas of charts for $\mathbb{RP}^n$.

3. Prove that the projection $\pi : \mathbb{R}^{n+1} \setminus \{O\} \to \mathbb{RP}^n$ is a smooth map.

Exercise 3 Let $F : M \to N$ a differentiable map between manifolds, with $\dim M = n + k \geq n = \dim N$. A point $p \in M$ is a *critical point* of F is $dF_p : T_pM \to T_{F(p)}N$ is not surjective. A *critical value* is the image of a critical point. A *regular value* is a point in $F(M)$ that is not a critical value. We denote by $\mathrm{Crit}(F) \subset M$ the set of critical points of F. Prove the following:

1. for every $a \in F(M)$ the set $M_a = F^{-1}(a) \setminus \mathrm{Crit}(F)$ is a submanifold of M of dimension k. In particular if $a \in N$ is regular, then $F^{-1}(a)$ is a submanifold of M of dimension k;

2. if $p \in M_a$ the tangent space of M_a in p coincides with the kernel of dF_p. In particular, if $N = \mathbb{R}$ and $F = f \in \mathcal{C}^\infty(M)$ then the tangent space of M_a in p is given by the vectors $v \in T_pM$ such that $v(f) = 0$.

(Hint: You can use exercise 1.)

Exercise 4 For each of the following pairs of vector fields V, W defined on $\mathbb{R}^3$, compute the Lie bracket $[V, W]$:

1. $V = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}$; $W = \frac{\partial}{\partial y}$;
2. $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$; $W = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$;
3. $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$; $W = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$.