Differential Geometry
Exercise Sheet 3

Exercise 1 Prove the following:

1. if $F : G_1 \to G_2$ is a homomorphism of Lie groups, then the rank of F is constant.
2. the kernel of F is a closed smooth submanifold and a Lie group with

$$\dim(\ker \mathrm{F}) = \dim G_1 - \mathrm{rank}\, F\,.$$

Exercise 2 Denote by R the additive Lie group of real numbers $\mathbb{R}$. Let $M = Gl(2, \mathbb{R}^2)$ and define an action of R on M by the formula

$$\theta(t, A) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A$$

with $A \in M$. Find the infinitesimal generator X and determine the orbits.

Exercise 3 Let $\mathbb{S}^3 = \{(x^1, x^2, x^3, x^4) \mid \sum_{i=1}^4 (x^i)^2 = 1\}$. Let the vector fields be given by:

- $X = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4}$
- $Y = -x^3 \frac{\partial}{\partial x^1} - x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^4}$
- $Z = -x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}$.

at the point $(x^1, x^2, x^3, x^4) \in \mathbb{S}^3$. Prove the following:

1. the vector fields are independent;
2. the vector fields are tangent to $\mathbb{S}^3$;
3. they are C^∞-vector fields.

Exercise 4 For each of the following pairs of vector fields V, W defined on $\mathbb{R}^3$, compute the Lie bracket $[V, W]$:

1. $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$; $W = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$;
2. $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$; $W = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$.

Exercise 5 Let $M_1, \ldots, M_k$ be smooth manifolds and let $\pi_i : M_1 \times \ldots \times M_k \to M_i$ be the projection on the i-th factor. Show that for any choices of $p_i \in M_i$ with $i = 1 \ldots k$ the map

$$\alpha : T_{(p_1,\ldots,p_k)}(M_1 \times \ldots \times M_k) \to T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k$$

defined by

$$\alpha(X) = (\pi_{1\star}X, \ldots, \pi_{k\star}X) .$$

is an isomorphism.