Differential Geometry
Exercise Sheet 4

Exercise 1 Let S be the unit sphere in $\mathbb{R}^3$. Determine the the coefficients g_{ij} of the first fundamental form (i.e. the Riemannian metric) on the domain of the coordinate chart in each of the following cases

1. Using spherical coordinates $(x^1 = \theta, x^2 = \phi)$ for S representing longitude and latitude;
2. Using stereographic coordinates (x^1, x^2).

Exercise 2 Let V be a vector space with $\dim(V) > 1$. Show that when $r = 2$ we have $\bigotimes^r(V) = \Lambda^r(V) \oplus \mathrm{Sym}^r(V)$, but that this is false whenever $r > 2$.

Exercise 3 Recall that a vector bundle on S is defined, in this class, to be a submanifold of $S \times \mathbb{R}^n$ such that the fibers of the projection to S are vector subspaces, and that a map of vector bundles is a map of the underlying manifolds that is linear on the fibers.

Use the inverse function theorem to show that every vector bundle is locally isomorphic to the trivial vector bundle. (Equivalently, if E is a rank r vector bundle on S, and p is a point in S, then on some neighborhood U of p, there is a set of r sections of E over U which form a basis of E_q for each $q \in U$. This is called a frame for E over U.)

Exercise 4 Let $R \subset S \times S$ be an equivalence relation satisfying the conditions of the quotient theorem, let $\pi_i : R \to S$ be the two projections, and let E be a vector bundle on S. Suppose $\phi : \pi_1^* E \to \pi_2^* E$ is an isomorphism of vector bundles such that for all triples of points $x, y, z \in S$, $\phi_{(x,z)} : E_x \to E_z$ is equal to the composition $\phi_{(y,z)} \circ \phi_{(x,y)}$. Let $\sim$ be the equivalence relation on (the total space of) E such that $(x, e_x) \sim (y, e_y)$ if and only if $(x, y) \in R$ and $\phi_{(x,y)}(e_x) = e_y$.

1. Show that $\sim$ is really an equivalence relation.
2. Show that the set $\hat{R} \subset E \times E$ defined by $\sim$ (i.e. $(e, e') \in \hat{R}$ iff $e \sim e'$) satisfies the conditions of the quotient theorem, and conclude that $E/\hat{R}$ is a manifold.

Though you do not need to prove it, in fact $E/\hat{R}$ is a vector bundle over S/R.

Exercise 5 Let ω be a covector field on a manifold S, and let X be a vector field on S. Suggest a definition of the Lie derivative $L_X\omega$, and explain why this is a good choice of the definition. In particular, justify your choice of a minus sign or a plus sign.