Differential Geometry
Exercise Sheet 5

Exercise 1 Let $\mathcal{A} : \mathcal{T}^r(V) \to \mathcal{T}^r(V)$ be the alternating mapping, prove the following:

1. $\mathcal{A}(\phi \otimes \psi \otimes \theta) = \mathcal{A}(\mathcal{A}(\phi \otimes \psi) \otimes \theta)$;
2. $\mathcal{A}(\mathcal{A}(\phi \otimes \psi) \otimes \theta) = \mathcal{A}(\phi \otimes \mathcal{A}(\psi \otimes \theta))$;
3. Use these facts to deduce that the exterior product is associative.

Exercise 2 Let $\phi_1, \ldots, \phi_r$ be elements of $V^\star = \Lambda^1 V$. Show that they are linearly indipendent if and only if $\phi_1 \wedge \ldots \wedge \phi_r \neq 0$

Exercise 3 Prove that the volume of the parallelepiped of $\mathbb{R}^3$ whose vertex is at the origin and whose sides from this vertex are the vectors $v_i = (x_i^1, x_i^2, x_i^3)$ with $i = 1, 2, 3$ is in fact the determinant of the matrix (x_i^j).

Exercise 4 Compute the expression for Ω on $\mathbb{S}^2$ (with the induced metric of $\mathbb{R}^3$) in terms of the coordinates given by:

1. stereographic projection;
2. spherical coordinates (ρ, θ, ϕ) with $\rho = 1$.

Exercise 4 Show that if D is a domain of integration on a manifold, then

$$\int_D f = \int_{\overline{D}} f = \int_{\mathring{D}} f \, .$$

Exercise 5 Using (Boothy Chapter VI, Remark 2.7) integrate on $M = \mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$, the function f giving the distance of a point on M from the plane $x^3 = -1$. Argue that we may use as D_1 and D_2 the upper and lower hemispheres.