Differential Geometry
Exercise Sheet 6

Exercise 1 Suppose that Δ is a distribution on M and suppose that $\mathcal{J}$ is the collection of 1-forms ϕ on M which vanish on Δ. Prove that Δ is in involution if and only if $d\mathcal{J} = \{d\phi \mid \phi \in \mathcal{J}\}$ is in the ideal generated by $\mathcal{J}$.

Exercise 2 Let $M = \mathbb{R}^3$ and determine which of the following are closed and which are exact:

1. $\phi = yzdx + xzdy + xydz;$
2. $\phi = xdx + x^2y^2dy + yzdz;$
3. $\theta = 2xy^2dx \wedge dy + zdy \wedge dz.$

Exercise 3 Show that the following is true for every $\phi \in \Lambda^r(M)$:

$$d\phi(X_1, \dots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i-1} X_i \phi(X_1, \dots, \widehat{X}_i, \dots, X_{r+1}) + \sum_{i<j} (-1)^{i+j} \phi([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{r+1}) \, .$$

Exercise 4 Let $\mathbb{S}^2 = \partial \bar{B}_3$ and $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$. Calculate $\int_{\mathbb{S}^2} \omega$ in two ways:

1. using Stokes Theorem
2. without using Stokes Theorem

Exercise 5 Suppose that D is compact regular domain in $\mathbb{R}^2$, and P, Q are smooth real functions on D. Prove the following (Green's theorem):

$$\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \; dxdy = \int_{\partial D} Pdx + Qdy \, .$$