Differential Geometry
Exercise Sheet 7

Exercise 1 Let S^2 be the unit sphere, oriented by its outward normal. Show that if two unit-speed parametrized curves α and β in S^2 have the same signed geodesic curvature for all s, then they are equal up to a rotation of S^2. (Hint: show that the Darboux frames agree up to a rotation).

Exercise 2 Given the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \qquad s \in \mathbb{R},$$

where $c^2 = a^2 + b^2$.

1. Show that the parameter s is arclength.
2. Determine the curvature and torsion of α.
3. Determine the osculating plane of α.

Exercise 3 Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve (not necessarily by arclength) and let $\beta : J \to \mathbb{R}^3$ be a reparametrization of α by arclength $s = s(t)$. Let $t = t(s)$ be the inverse function of s and set $d\alpha/dt = \alpha'$, $d^2\alpha/dt^2 = \alpha''$, etc. Prove that

1. $dt/ds = 1/|\alpha'|$ and $d^2t/ds^2 = -\langle\alpha', \alpha''\rangle/|\alpha'|^4$.
2. The curvature of α at $t \in I$ is
$$\kappa(t) = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}.$$
3. The torsion of α at $t \in I$ is
$$\tau(t) = -\frac{\langle\alpha' \times \alpha'', \alpha'''\rangle}{|\alpha' \times \alpha''|^2}.$$

Exercise 4 Let

$$\psi(x_1, x_2) = (x_1 \cos x_2, x_1 \sin x_2, x_2)$$

be a parametrization of the helicoid.

1. Compute the mean curvature H and Gauss curvature K of the helicoid.
2. Find the asympotic curves and the lines of curvature of the helicoid.