Differential Geometry
Exercise Sheet 8

Exercise 1 Justify why the following surfaces are not pairwise locally isometric:

1. Sphere
2. Cylinder
3. Saddle $z = x^2 - y^2$

Exercise 2 Define a connection on $\mathbb{R}^3$ by setting (in standard coordinates)

$$\Gamma^3_{12} = \Gamma^1_{23} = \Gamma^2_{31} = 1$$

$$\Gamma^3_{21} = \Gamma^1_{32} = \Gamma^2_{13} = -1.$$

with all other connection coefficients equal to zero. Show that this connection is compatible with the Euclidean metric and has the same geodesics as the Euclidean connection, but is not symmetric.

Exercise 3 Let (M, g) be a Riemannian manifold, and ∇ the Levi-Civita connection. Show that the following formula holds for every smooth 1-form η on M:

$$d\eta(X, Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X)$$

Exercise 4 Let $M \subset \mathbb{R}^3$ be a surface of revolution, parametrized by

$$\phi(\theta, t) = (a(t)\cos\theta, a(t)\sin(\theta), b(t))$$

It will simplify computations if we assume that the curve $\gamma(t) = (a(t), b(t))$ is parametrized at unit speed.

1. Compute the Christoffel symbols of the induced metric in (θ, t) coordinates.
2. Show that each "meridian" $\{\theta = \theta_0\}$ is a geodesic on M.
3. Determine necessary and sufficient conditions for a "lattitude circle" $\{t = t_0\}$ to be a geodesic.

Exercise 5 Let (M, g) be a Riemannian manifold. The gradient grad f of a smooth function f is the vector field dual to the one-form df. If f is a smooth function on M such that $|df| = 1$, show that the integral curves of grad f are geodesics.