Differential Geometry
Exercise Sheet 9

Exercise 1 Let $\mathbb{R}^+$ with the metric $||t_h|| = h^{-1}(|t|)$ for every $h \in \mathbb{R}^+$ and $t \in T_h\mathbb{R}^+$ where we identified $T_h\mathbb{R}^+$ with $\mathbb{R}$ as usual. Show that $\exp_h : T_h\mathbb{R}^+ \to \mathbb{R}^+$ is given by the formula $\exp_h(t) = he^t$.

Exercise 2 Show that $\mathbb{R}^n$ and $\mathbb{S}^n_R$ are complete with respect to their standard Riemannian metrics.

Exercise 3 A subset U of a Riemannian manifold M is said to be convex if for each $p, q \in U$, there is a unique (in M) minimizing geodesic from p to q lying entirely in U. Show that every point has a convex neighborhood, as follows:

1. Let $p \in M$ be fixed, and let W be a uniformly normal neighborhood of p. For $\varepsilon > 0$ small enough that $B_{2\varepsilon}(p) \subset W$, define a subset $W_\varepsilon \subset TM \times \mathbb{R}$ by
$$W_\varepsilon = \left\{(q, V, t) \in TM \times \mathbb{R} : q \in B_\varepsilon(p), V \in T_qM, |V| = 1, |t| < 2\varepsilon\right\}.$$
Define $f : W_\varepsilon \to \mathbb{R}$ by
$$f(q, V, t) = d(\exp_q(tV), p)^2.$$
Show that f is smooth. [Hint: Use normal coordinates centered at p.]
2. Show that if ε is chosen small enough, then $\frac{\partial^2 f}{\partial t^2} > 0$ on W_ε. [Hint: Compute $f(p, V, t)$ explicitly and use continuity.]

Exercise 4 Let $M \subset \mathbb{R}^3$ be a compact, orientable, embedded 2-manifold with the induced metric.

1. Show that M cannot have $K \leq 0$ everywhere. [Hint: Look at a point where the distance from the origin takes a maximum.]
2. Show that M cannot have $K \geq 0$ everywhere unless $\chi(M) > 0$.

Exercise 5 Determine the Gaussian curvature K of the surface $S \subset \mathbb{R}^3$ of equation $x^2 + y^2 = (\cosh z)^2$, and calculate:
$$\int_S K d\mu \,.$$